Algebraic Simplex
Active Learning – Module 4

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Time required for the module: 50 Min.

Reading

✦ *Most of the introductory/intermediate textbooks in Operations Research have a chapter that covers the fundamentals of the simplex algorithm. The students should read this chapter before coming to class. Two suggestions:*

 ✻ *Chapter 3 of* Operations Research, by Hamdy A. Taha, Prentice Hall , 7th Edition

 ✻ *Chapter 4 of Introduction to Operations Research by Hillier and Liberman, 7th Edition*

✦ *The student should also read the self-contained module on the standard form of an LP problem. This module can be downloaded from: http://enpc2675.eas.asu.edu/lobos/iie476/stdform.doc*

✦ *While not required, it is recommended that the active learning module on the Simplex Tableau is read before reading this module*

Lecture objectives

✦ *At the end of the lecture each student should be able to:*

  ✱ *Explain the logic behind the algorithmic steps of the Simplex Algorithm*

  ✱ *Find the optimal solution of a Linear Programming problem by using the algebraic approach of the Simplex Algorithm*

Time Management

✦ *Introduction 3 minutes*

✦ *RAT 5 minutes*

✦ *Lecturing on the mechanics of the Simplex Algorithm 20 minutes*

✦ *Team exercise 7 minutes*

✦ *Lecturing on Simplex algorithm 10 minutes*

✦ *Wrap up 5 minutes*

✦ *Total lecture time 50 minutes*

Individual Exercise
Readiness Assessment test(3 minutes)

✦*Setup the following problem as a standard LP Problem:*

Minimize z = 3x + 5y – 3z

Subject to:

$-3x - 2y + 5z \geq -20$

$x + y \leq 10$

$z \geq 4$

✦*At the end of the exercise turn in your solution*

Solution

Minimize z = 3x + 5y – 3z

Subject to:

$$+3x + 2y - 5z \leq 20$$

$$x + y \leq 10$$

$$y \geq 4$$

Maximize -z = -3x - 5y + 3z

Subject to:

$$+3x + 2y - 5z + S_1 = 20$$

$$x + y + S_2 = 10$$

$$Y - S_3 = 4$$

Solution of a Linear Programming Problem

✦ *Once a problem can be expressed as a linear programming problem the next step is to find its optimal solution*

✦ *For trivial 2 and 3 decision variable problems their optimal solution can be found graphically*

✦ *For more complex problems the simplex algorithm is traditionally used. → In this lecture we use illustrate the algebraic operations of the Simplex Algorithm*

✦ *In order to use the simplex algorithm it is necessary to:*

　∗ *Express the problem in a standard form*
　∗ *Represent the objective function as an equation*

$$z - \sum_j c_j \chi_j = 0;$$

where z is the value of the objective function and the x_j's the decision variables

Standard Form

✦ *We define the standard form of a linear programming problem as:*

✱ *One whose objective function is maximization*

✱ *One whose constraints are expressed as equations and whose right hand side is non-negative (after the introduction of slack and surplus variables)*

✱ *One in which all the decision variables are non-negative*

Example: Setting up the Reddy-Mikks Problem

✦ For Instance the Reddy-Mikks Problem:

Maximize $z = 3X_E + 2X_I$

Subject to:

$X_E + 2X_I \leq 6$ (1)

$2X_E + X_I \leq 8$ (2)

$-X_E + X_I \leq 1$ (3)

$X_I \leq 2$ (4)

$X_E, X_{I,} \geq 0$

✦ Is expressed as:

$z - 3X_E - 2X_I - 0S_1 - 0S_2 - 0S_3 - 0S_4$

Subject to:

$X_E + 2X_I + S_1 \qquad\qquad = 6$ (1)

$2X_E + X_I \qquad + S_2 \qquad = 8$ (2)

$-X_E + X_I \qquad\qquad + S_3 = 1$ (3)

$X_I \qquad\qquad\qquad + S_4 = 2$ (4)

$X_E, X_{I,}, S_{1,}, S_{2,}, S_{3,}, S_4 \geq 0$

Maximization is implied by the (+1)z since it is the objective function of the standard form.

Conceptual Outline of the Steps of the Simplex Algorithm

✦ **Step 0:** Using the standard form determine a starting basic feasible solution by setting $n-m$ non-basic variables to zero.

✦ **Step 1**: Select an entering variable from among the current non-basic variables, which gives the largest per-unit improvement in the value of the objective function.
If none exists stop; the current basic solution is optimal. Otherwise go to Step 2.

✦ **Step 2**: Select a leaving variable from among the current basic variables that must now be set to zero (become non-basic) when the entering variable becomes basic.

✦ **Step 3**: Determine the new basic solution by making the entering variable, basic; and the leaving variable, non-basic, and return to Step 1.
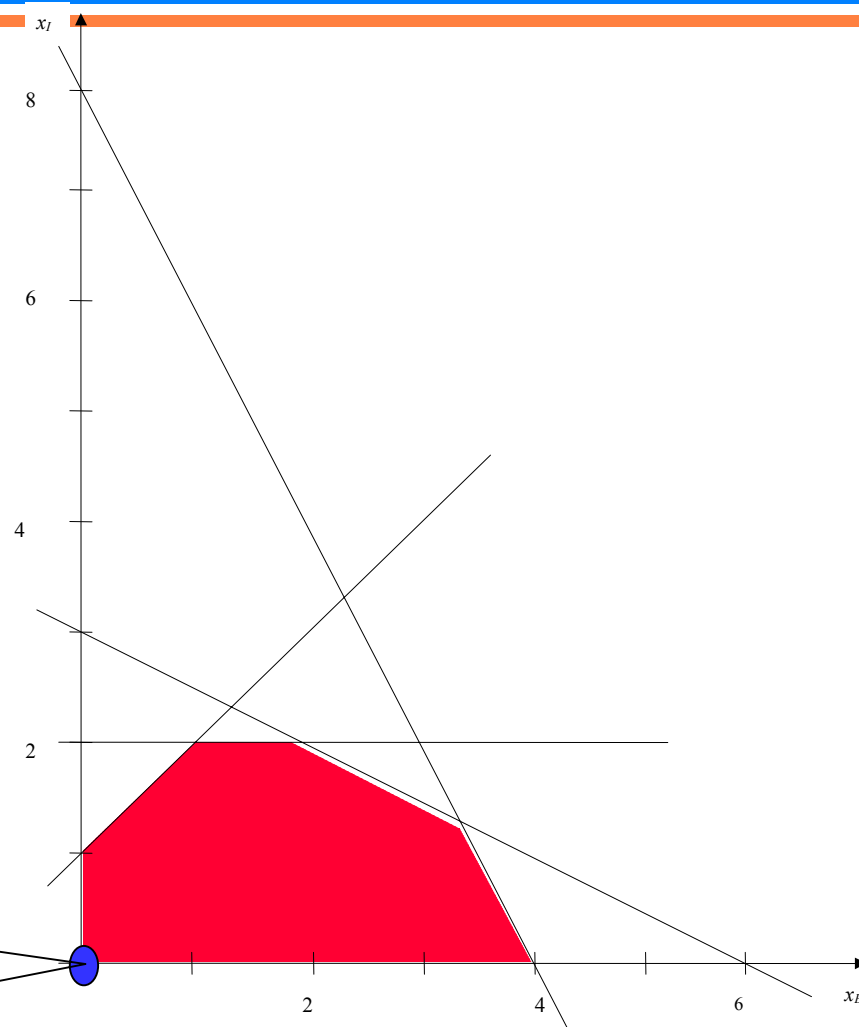
Example 1: Reddy-Mikks Problem

✦ **Step 0**.- Since we have four equations representing the constraints we can have at most four linearly independent variables to get the solution of the system of equations.

✦ We set X_E, X_I, to zero (non-basic variables) and solve for S_1, S_2, S_3, S_4 (basic variables). Notice that we easily get a feasible solution. In particular a *basic feasible solution*.

$$z - 3X_E - 2X_I - 0S_1 - 0S_2 - 0S_3 - 0S_4$$
$$X_E + 2X_I + S_1 = 6 \ (1)$$
$$2X_E + X_I + S_2 = 8 \ (2)$$
$$-X_E + X_I + S_3 = 1 \ (3)$$
$$X_I + S_4 = 2 \ (4)$$
$$X_E, X_I, S_1, S_2, S_3, S_4 \geq 0$$

$$z \qquad\qquad\qquad - 0S_1 - 0S_2 - 0S_3 - 0S_4 = 0 \ (0)$$
$$+ S_1 = 6 \ (1)$$
$$+ S_2 = 8 \ (2)$$
$$+ S_3 = 1 \ (3)$$
$$+ S_4 = 2 \ (4)$$
$$X_E, X_I, S_1, S_2, S_3, S_4 \geq 0$$

Example 1: Reddy-Mikks Problem

Current
Solution:
$X_E = 0$
$X_I = 0$
$S_1 = 6$
$S_2 = 8$
$S_3 = 1$
$S_4 = 2$

Example 1: Reddy-Mikks Problem (Cont.)

✦ **Step 1**.- Select an entering variable from among the current non-basic variables which can improve the value of the objective function.
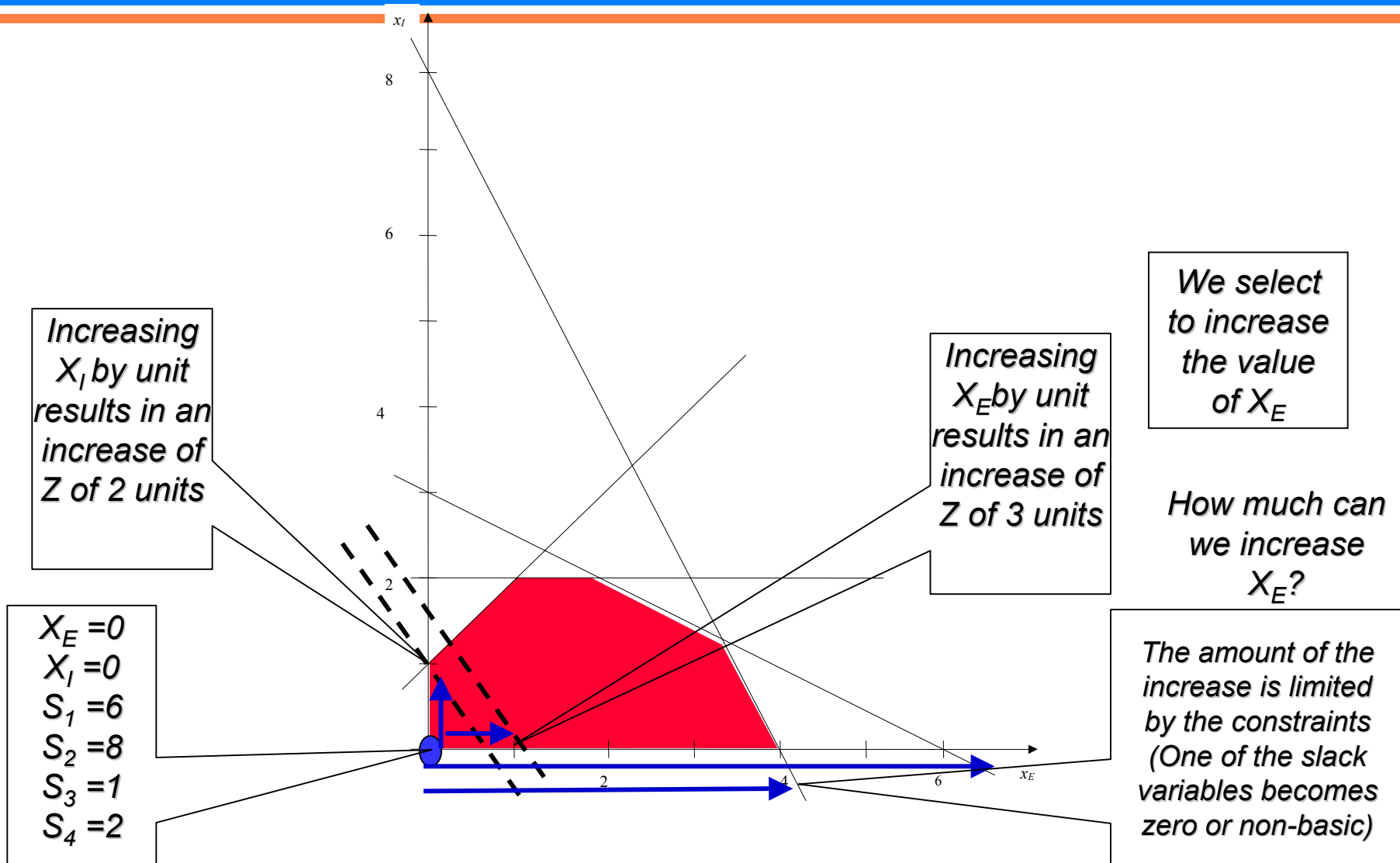
  Current non-basic variables: X_E , X_I,

Objective function: z -$3X_E$ - $2X_I$ + $0S_1$ + $0S_2$ + $0S_3$+ $0S_4$

✦ Thus increasing the value of either of the non-basic variables will improve the current solution (z=0). Since the per-unit contribution of X_E (3) is higher than X_I (2) we select X_E as the entering (new basic) variable.

(Note that these per-unit contributions are the partial derivatives of the Objective Function)

Example 1: Reddy-Mikks Problem

Increasing X_I by unit results in an increase of Z of 2 units

Increasing X_E by unit results in an increase of Z of 3 units

We select to increase the value of X_E

How much can we increase X_E?

$X_E = 0$
$X_I = 0$
$S_1 = 6$
$S_2 = 8$
$S_3 = 1$
$S_4 = 2$

The amount of the increase is limited by the constraints (One of the slack variables becomes zero or non-basic)

Example 1: Reddy-Mikks Problem (Cont.)

✦ **Step 2**: Select a leaving variable from among the current basic variables that must be set to zero (become non-basic) when the entering variable becomes basic.

✦ Since we want to increase the value of X_E as much as possible we increase it until one of the slack variables becomes zero (it then becomes non-basic). In this case S_2 (when the value of X_E is increased to 4) is the one that that becomes non-basic. (To see this use the original equations and ask how big X_E can be before the slack variable is driven to zero in each equation.)

✦ Note that by doing a single pair wise exchange of basic(leaving) and non-basic (entering) variables result in moving to an adjacent corner)

Example 1: Reddy-Mikks Problem (Cont.)

✦ **Step 3**: Determine the new basic solution by making the entering variable, basic; and the leaving variable, non-basic.

$z - 3X_E - 2X_I - 0S_1 - 0S_2 - 0S_3 - 0S_4 = 0 \ (0)$

$X_E + 2X_I + S_1 \qquad\qquad = 6 \ (1)$

$2X_E + X_I \qquad + S_2 \qquad\qquad = 8 \ (2)$

$-X_E + X_I \qquad\qquad + S_3 \qquad = 1 \ (3)$

$\qquad X_I \qquad\qquad\qquad + S_4 \quad = 2 \ (4)$

$X_E, X_I, S_1, S_2, S_3, S_4 \geq 0$

We need to solve this system of equations → Use elementary row operations to the get the proper form. Apply this to all the variables
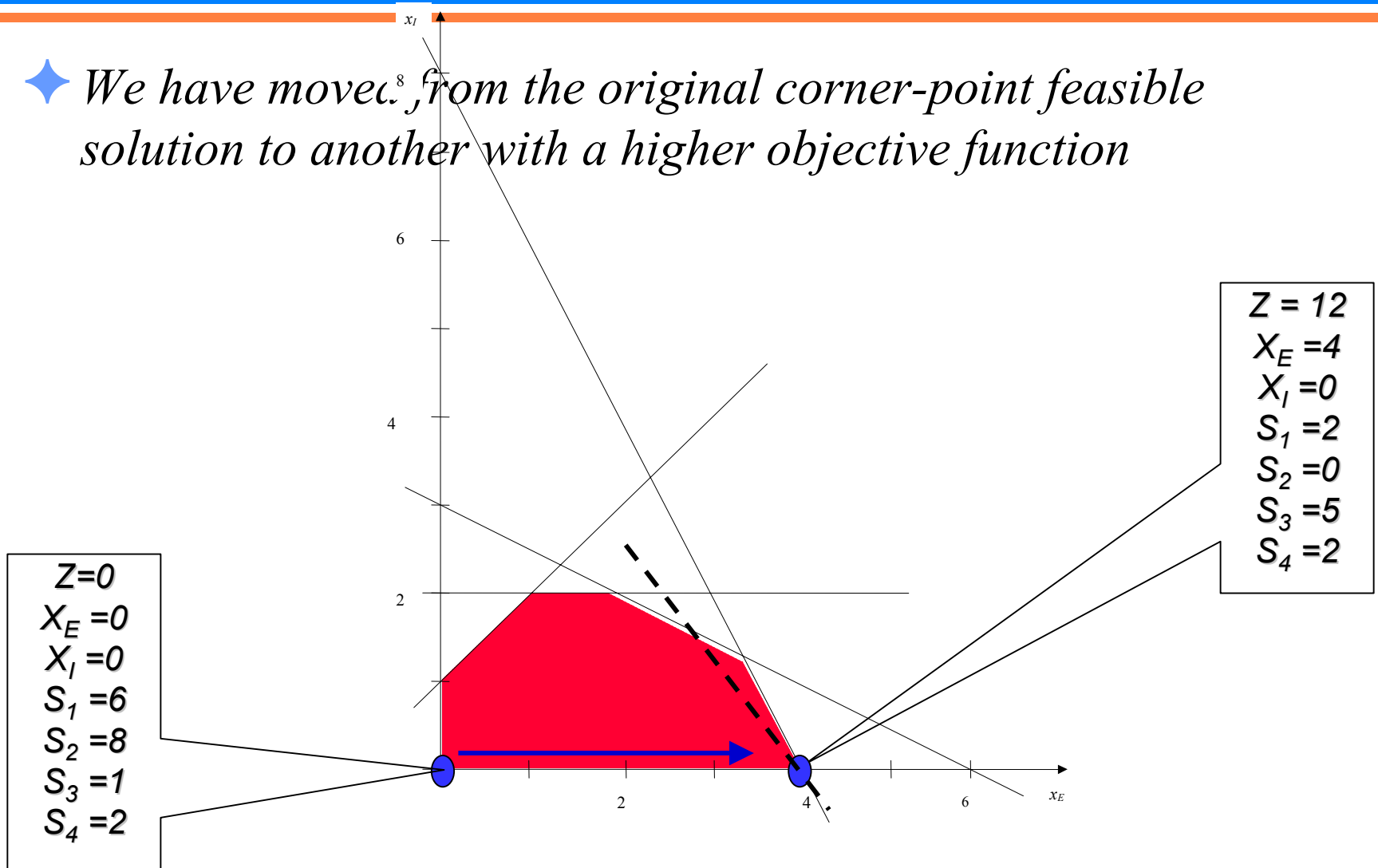
$z - 0X_E - 0.5X_I - 0S_1 + 1.5S_2 - 0S_3 - 0S_4 = 12 \ (0)$

$1.5X_I + S_1 - 0.5S_2 \qquad\qquad = 2 \ (1)$

$X_E + 0.5X_I \qquad + 0.5S_2 \qquad\qquad = 4 \ (2)$

$1.5 X_I \qquad + 0.5S_2 + S_3 \qquad = 5 \ (3)$

$X_I \qquad\qquad\qquad + S_4 = 2 \ (4)$

$X_E, X_I, S_1, S_2, S_3, S_4 \geq 0$

✦ We return to step 1.

Example 1: Reddy-Mikks Problem

✦ *We have moved from the original corner-point feasible solution to another with a higher objective function*

x_I

x_E

6

4

2

2   4   6

8

Z = 12
X_E =4
X_I =0
S_1 =2
S_2 =0
S_3 =5
S_4 =2

Z=0
X_E =0
X_I =0
S_1 =6
S_2 =8
S_3 =1
S_4 =2

Example 1: Reddy-Mikks Problem

✦ **Step 1**.- Select an entering variable from among the current non basic variables which gives the greatest per-unit improvement in the value of the objective function.
Current non-basic variables: X_I, S_2

We select X_I since it is the only one that will improve the current value of objective function

$$z - 0X_E - 0.5X_I - 0S_1 + 1.5S_2 - 0S_3 - 0S_4 = 12 \quad (0)$$
$$1.5X_I + S_1 - 0.5S_2 = 2 \quad (1)$$
$$X_E + 0.5X_I + 0.5S_2 = 4 \quad (2)$$
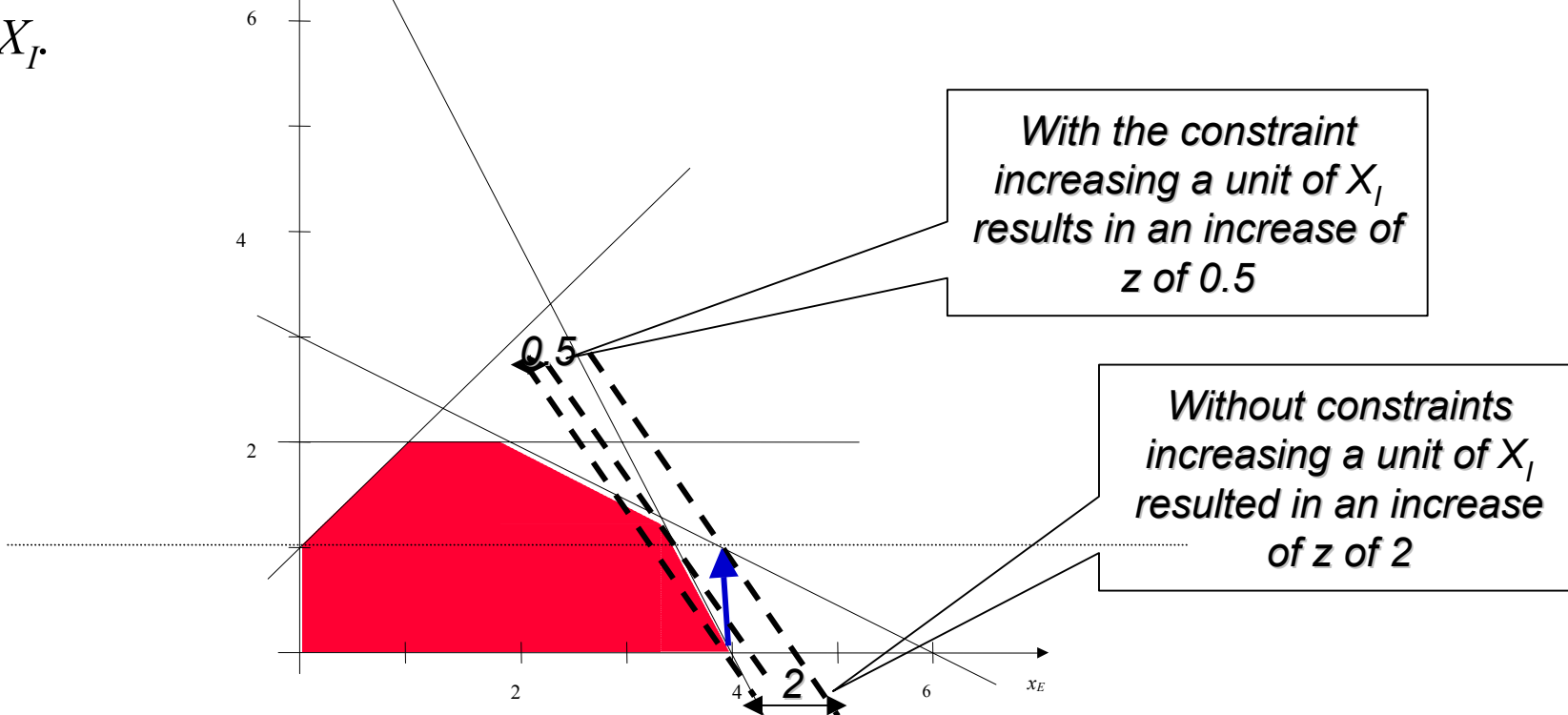$$1.5\,X_I + 0.5S_2 + S_3 = 5 \quad (3)$$
$$X_I + S_4 = 2 \quad (4)$$
$$X_E,\ X_I,\ S_1,\ S_2,\ S_3,\ S_4 \geq 0$$

Team Exercise (5 min)

✦ *In Step 1 of the previous iteration we decided to use the variable X_E as the entering variable because of its marginal contribution to the improvement of the objective function (3 units) over the marginal contribution of the variable X_I (2 units).*

✦ *In the current solution the marginal contribution of X_I to the improvement of the objective function is only 0.5 units, why?*

✦ *Give a plausible explanation to the difference in the marginal contribution of X_I*

   ∗ *In words*

   ∗ *Graphically*

✦ *The instructor will randomly select the members of two teams to give their solutions*

Solution

✦ *Since X_E is bound by a constraint that is a function of X_E and X_I, if we increase X_I we need to reduce X_E by an amount dictated by the constraint. Thus, increasing X_I by one unit will not result in an increase of z by the original marginal contribution of X_I.*

x_I

8

6

4

2

0.5

With the constraint increasing a unit of X_I results in an increase of z of 0.5

Without constraints increasing a unit of X_I resulted in an increase of z of 2

2

2

4

6

x_E

Example 1: Reddy-Mikks Problem (Cont.)

✦ **Step 1**.- Select an entering variable from among the current non basic variables which gives the greatest per-unit improvement in the value of the objective function.
Current non-basic variables: X_1, S_2

  We select X_1 since it is the only one that will improve the current value of objective function

✦ **Step 2**: Select a leaving variable from among the current basic variables that must be set to zero (become non-basic) when the entering variable becomes basic.

Since we want to increase the value of X_1 as much as possible we do that until one of the current basic variable becomes zero (non-basic). In this case S_1 (when the value of X_1 is increased to 4/3) is the one that that becomes non-basic.

Example 1: Reddy-Mikks Problem (Cont.)

✦ **Step 3**: Determine the new basic solution by making the entering variable basic; and the leaving variable, non-basic.

$z - 0X_E - 0.5X_I$ ⬛ $0S_3 - 0S_4 = 12(0)$
$1.5X_I \qquad = 2\ (1)$
$X_E + 0.5X_I \qquad = 4\ (2)$
$1.5\ X_I \qquad + S_3 \quad = 5\ (3)$
$X_I \qquad\qquad + S_4 = 2\ (4)$
$X_E,\ X_I,\ S_1,\ S_2,\ S_3,\ S_4 \geq 0$

$z - 0X_E + 0X_I + 0.33S_1 + 1.33S_2 - 0S_3 - 0S_4 = 12.66\ (0)$
$X_I + 0.66S_1 - 0.33S_2 \qquad = 1.33\ (1)$
$X_E \qquad -0.33S_1 + 0.66S_2 \qquad = 3.33\ (2)$
$-1.0S_1 + 1.0S_2 + S_3 \qquad = 3 \qquad (3)$
$-0.66S_1 + 0.33S_2 \qquad + S_4 = 0.66\ (4)$
$X_E,\ X_I,\ S_1,\ S_2,\ S_3,\ S_4 \geq 0$

✦ When we now return to step 1 no further improvement is indicated so we stop; this is the optimum (maximum).

Summary

- ✦ *The simplex starts from a basic feasible solution*

- ✦ *By exchanging one basic for one non-basic variables it moves from one corner point solution to an adjacent, more promising, corner point solution*

- ✦ *It continues with this process until it reaches the optimal solution of the LP problem*

Assignment

✦ *Find the optimal solution of the* Report Strategy Problem *by using the algebraic simplex method. As part of the assignment show the corner points visited in the 2D graph of the problem.*