

LP Definition and Introduction to Graphical Solution Active Learning – Module 2

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Background Material

- ◆ *Almost any introductory Operations Research book has a chapter that covers the graphical solution of a Linear Programming problem. The students should read this chapter before coming to class.*
- ◆ *Two suggestions:*
 - * *Chapter 2 (Sections 2.1 and 2.2) of Operations Research, Seventh Edition, by Hamdy A. Taha, Prentice Hall*
 - * *Chapter 3 (Sections 3.1, 3.2, 3.3 and 3.4) of Introduction to Operations Research by Hillier and Liberman*

Lecture objectives

◆ *At the end of the lecture, each student should be able to:*

- * Construct the 2D graphical solution of the LP problem*
- * Interpret the 2D graphical solution of the LP problem*
- * Explain the concepts of solution, feasible solution, feasible region, and corner point solution*

** Expected Lecture Time = 50 min*

Time Management

<i>Introduction</i>	<i>3 minutes</i>
<i>RAT</i>	<i>5 minutes</i>
<i>Lecturing on LP Assumptions</i>	<i>10 minutes</i>
<i>Team Exercise</i>	<i>5 minutes</i>
<i>Lecturing on Graphical Solutions</i>	<i>18 minutes</i>
<i>Team Exercise</i>	<i>5 minutes</i>
<i>Wrap-Up</i>	<i>4 minutes</i>
<i>Total Lecture Time</i>	<i>50 minutes</i>

From a previous module (Report Strategy Problem)

- ◆ Suppose that team members A and C have decided to work together in their tasks. They have decided that their skills are complementary and, by working together in both of the tasks assigned to them, they can do a better job than if they work independently. A has very good writing speed, he can write 4 pages per hour, but his rate of grammatical errors is very high (3 errors per page). On the other hand, C is slower (2.5 pages per hour), but his error rate is very low (1 error per page). They want to write a report that will earn them the highest grade.

They have discovered that the grade given by the professor is very much influenced by the number of pages of the written assignments (the more pages, the higher the grade assigned). However, they have also discovered that for every five errors in grammar, he deducts the equivalent of one written page, and also that the maximum number of mistakes the instructor will tolerate in an assignment is 80. Help this team of students determine the best strategy (you need to define the meaning of this) given the total time constraint available to work in the project. The total time allocated to both of them is 8 hours. Also, because of previous commitments, C cannot work more than 6 hours on the project

Team exercise

Readiness Assessment test(3 minutes)

- ◆ *For the report strategy problem:*
 - * *Define the decision variables and the values they can take*
 - * *Determine the constraints (for example, define the scarce resources) on the values that the decision variables can take*
 - * *Determine the objective function*
- ◆ *At the end of the exercise, each team should turn in the individual and the team solutions in a single packet to be graded*
- ◆ *The instructor may select a member of the team to present the team's solution to the rest of the class*

Solution

◆ The following is a solution to the problem (there may be more than one correct solution)

1.- Determination of the decision variables

x_A = # of hours that student A should spend in preparing the report

x_C = # of hours that student C should spend in preparing the report

2.- Determination of the objective function

Maximize grade on report (can you think of another objective function?)

Let z = grade on the report (effective number of pages)

Factors: number of pages, number of errors (**converted to page units**)

Total number of pages = (pages per hour produced by A) * (number of hours spent by A in the assignment) + (pages per hour produced by B) * (number of hours spent by B in the assignment) = $4(\text{pages/hr})X_A(\text{hrs}) + 2.5(\text{pages/hr})X_C(\text{hrs})$

Pages subtracted because of errors: Number of errors/5 (errors/page) = (pages/hour)(hrs) (errors/page)(1/5 pages/error) = $[4(\text{pages/hr})X_A(\text{hrs}) + 3(\text{errors/page}) + 2.5(\text{pages/hr})X_C(\text{hrs})1(\text{error/page})](0.2 \text{ page/error})$

Maximize $z = 4X_A + 2.5X_C - (2.4X_A + 0.5X_C) = 1.6X_A + 2.0X_C$

Solution (cont.)

2.- Determination of Constraints

Number of errors not to exceed (less than) 80

$$\begin{aligned} \text{Number of errors} &= 4(\text{pages/hr})X_A(\text{hrs}) + 3(\text{errors/page}) + \\ &2.5(\text{pages/hr})X_C(\text{hrs}) + 1(\text{error/page}) = 12X_A + 2.5X_C \leq 80 \end{aligned}$$

$$\text{Total number of Hours available to perform the task (8)} = X_A + X_C \leq 8$$

$$\text{Number of Hours available for C to perform the task (6)} = X_C \leq 6$$

Non-negative values for the decision variables: $X_A, X_C \geq 0$

Final formulation of the problem

$$\text{Maximize } z = 1.6X_A + 2.0X_C$$

Subject to:

$$12X_A + 2.5X_C \leq 80$$

$$X_A + X_C \leq 8$$

$$X_C \leq 6$$

$$X_A, X_C \geq 0$$

Definition of a Linear Programming Problem

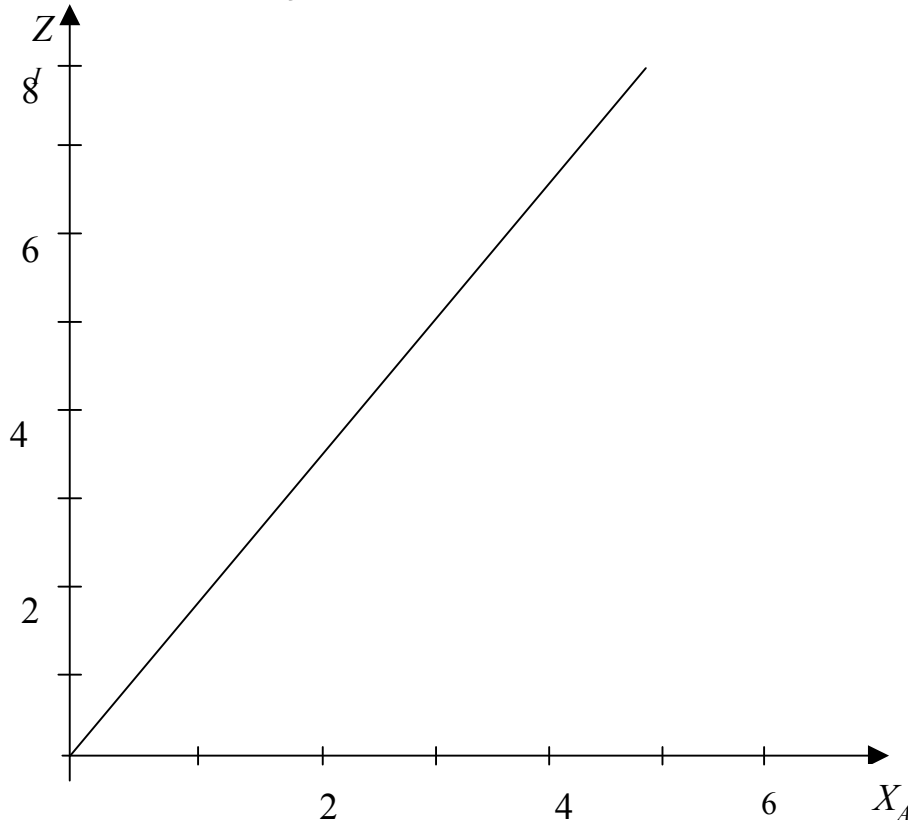
- ◆ *The previous model is a linear programming problem*
- ◆ *A linear programming problem (LP) is an optimization problem for which:*
 - 1) *The function to be maximized or minimized (called the objective function) is a linear function of the decision variables.*
 - 2) *The values of the decision variables must satisfy a set of constraints. Each constraint must be a linear equation or linear inequality.*
 - 3) *A sign restriction is associated with each variable. For any variable x_i , the sign restriction specifies that x_i must be non-negative.*

LP Assumptions

- ◆ When we use LP as an approximate representation of a real-life situation, the following assumptions are inherent:
 - * Proportionality. - The contribution of each decision variable to the objective or constraint is directly proportional to the value of the decision variable.
 - * Additivity. - The contribution to the objective function or constraint for any variable is independent of the values of the other decision variables, and the terms can be added together sensibly.
 - * Divisibility. - The decision variables are continuous and thus can take on fractional values.
 - * Deterministic. - All the parameters (objective function coefficients, right-hand side coefficients, left-hand side, or technology, coefficients) are known with certainty.

Examples of LP Assumptions

◆ *Example: The contribution of the X_A to the objective function can be represented as:*



Can you identify the assumptions of:

- *Proportionality*
- *Divisibility*
- *Deterministic*
- *Additivity?*

Team Exercise (three minutes)

- ◆ *Give two real-life examples in which one or more of the LP assumptions are violated*
- ◆ *The instructor will randomly select members of some teams to present their examples to the rest of the class*

The Windsor Glass Company Problem

(Hillier and Liberman)

The Windsor Glass Company is planning to launch two new products. Product 1 is an 8-foot glass door with aluminum framing and Product 2 a 4x6 foot double-hung wood-framed window

Aluminum frames are made in Plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products. Product 1 requires some of the production capacity in Plants 1 and 3, but none in Plant 2. Product 2 needs only Plants 2 and 3. The marketing division has concluded that the company could sell as much of either product as could be processed by these plants. The management of the company wants to determine what mixture of both products would be the most profitable. The following table provides the information available.

Plant	Production time per batch, hours		Production time available per week, hours
	Product		
	1	2	
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	\$3,000	\$5,000	

The Windsor Glass Company Problem

Formulation (Hillier and Liberman)

Formulation as a linear programming problem

Decision variables:

x_1 = Number of batches of product 1 produced per week

x_2 = Number of batches of product 2 produced per week

Objective function:

Maximize $z = 3 X_1 + 5X_2$ (in thousands of dollars)

Subject to:

$$x_1 \leq 4 \quad (\text{Production Available in Plant 1})$$

$$2x_2 \leq 12 \quad (\text{Production Available in Plant 2})$$

$$3x_1 + 2x_2 \leq 18 \quad (\text{Production Available in Plant 3})$$

$$x_1, x_2 \geq 0$$

Reddy Mikks Problem (Taha)

- ◆ *The Reddy Mikks company owns a small paint factory that produces both interior and exterior house paints for wholesale distribution. Two basic raw materials, A and B, are used to manufacture the paints.*
- ◆ *The maximum availability of A is 6 tons a day; that of B is 8 tons a day. The daily requirements of the raw materials per ton of interior and exterior paints are summarized in the following table.*

	Tons of Raw Material per Ton of Paint		Maximum Availability (tons)
	Exterior	Interior	
Raw Material A	1	2	6
Raw Material B	2	1	8

- ◆ *A market survey has established that the daily demand for the interior paint cannot exceed that of exterior paint by more than 1 ton. The survey also showed that the maximum demand for the interior paint is limited to 2 tons daily.*
- ◆ *The wholesale price per ton is \$3000 for exterior paint and \$2000 per interior paint. How much interior and exterior paint should the company produce daily to maximize gross income?*

Reddy Mikks Problem Formulation

Define:

X_E = Tons of exterior paint to be produced

X_I = Tons of interior paint to be produced

Thus, the LP formulation of the Reddy-Mikks Company is as follows:

Maximize $z = 3X_E + 2X_I$

Subject to:

$X_E + 2X_I \leq 6$ (1) (availability of raw material A)

$2X_E + X_I \leq 8$ (2) (availability of raw material B)

$-X_E + X_I \leq 1$ (3) (Restriction in production)

$X_I \leq 2$ (4) (Demand Restriction)

$X_E, X_I \geq 0$

Graphical Solution of an LP Problem

- ◆ *Used to solve LP problems with two (and sometimes three) decision variables*
- ◆ *Consists of two phases*
 - * *Finding the values of the decision variables for which all the constraints are met (feasible region of the solution space)*
 - * *Determining the optimal solution from all the points in the feasible region (from our knowledge of the nature of the optimal solution)*

Finding the Feasible Region (2D)

◆ *Steps*

- * Use the axis in a 2-dimensional graph to represent the values that the decision variables can take*
- * For each constraint, replace the inequalities with equations and graph the resulting straight line on the 2-dimensional graph*
- * For the inequality constraints, find the side (half-space) of the graph meeting the original conditions (evaluate whether the inequality is satisfied at the origin)*
- * Find the intersection of all feasible regions defined by all the constraints. The resulting region is the (overall) feasible region.*

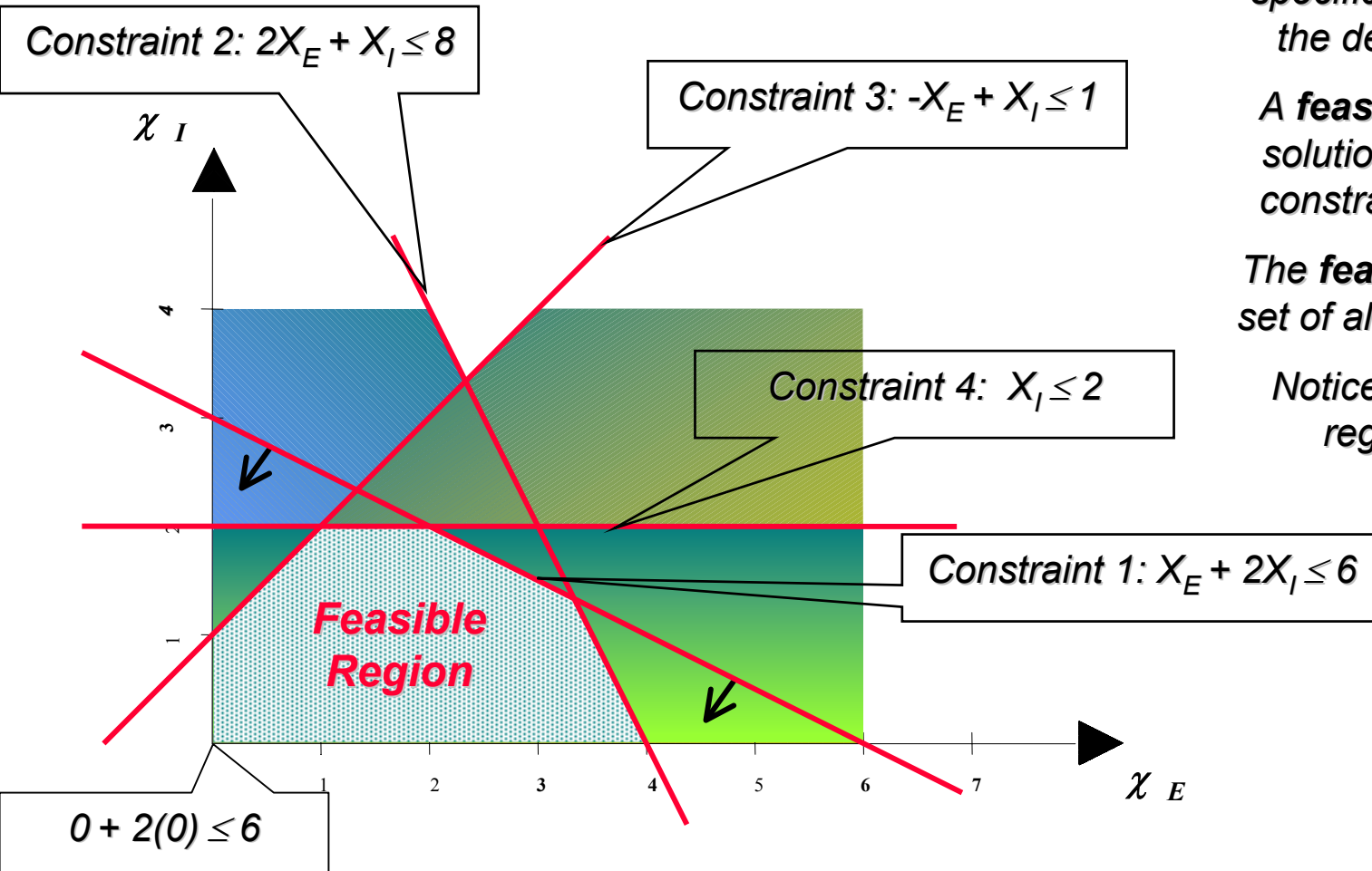
Graphical Solution of the Ready Mikks Problem

A **solution** is any specification of values for the decision variables.

A **feasible Solution** is a solution for which all the constraints are satisfied.

The **feasible region** is the set of all feasible solutions.

Notice that the feasible region is **convex**

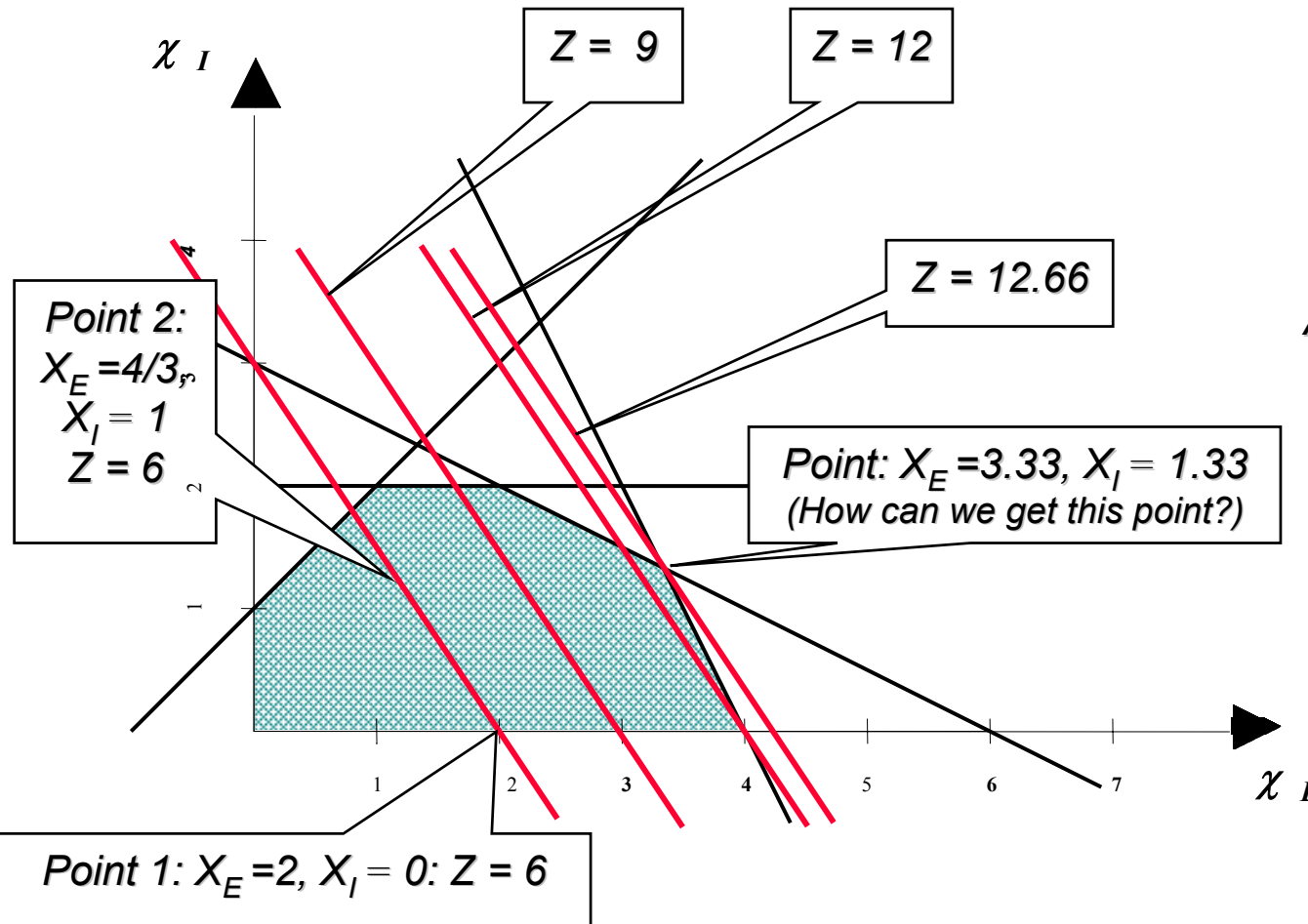


Finding the Optimal Solution

- ◆ *Determine the slope of the objective function (an infinite set of straight lines-isoclines)*
 - * *Select a convenient point in the feasible region*
 - * *Draw the corresponding straight line (a single isocline)*
- ◆ *Determine the direction of increase of the objective function (we are maximizing)*
 - * *Select a second point in the feasible region and simply evaluate the objective function at that point*
- ◆ *Follow the direction of increase until reaching the (corner) point beyond which any increase of the objective function would take you outside of the feasible region*

Graphical Solution of the Ready Mikks Problem

$$\text{Max } z = 3X_E + 2X_I$$



An **Optimal Solution** is a feasible solution that has the most favorable value of the objective function.

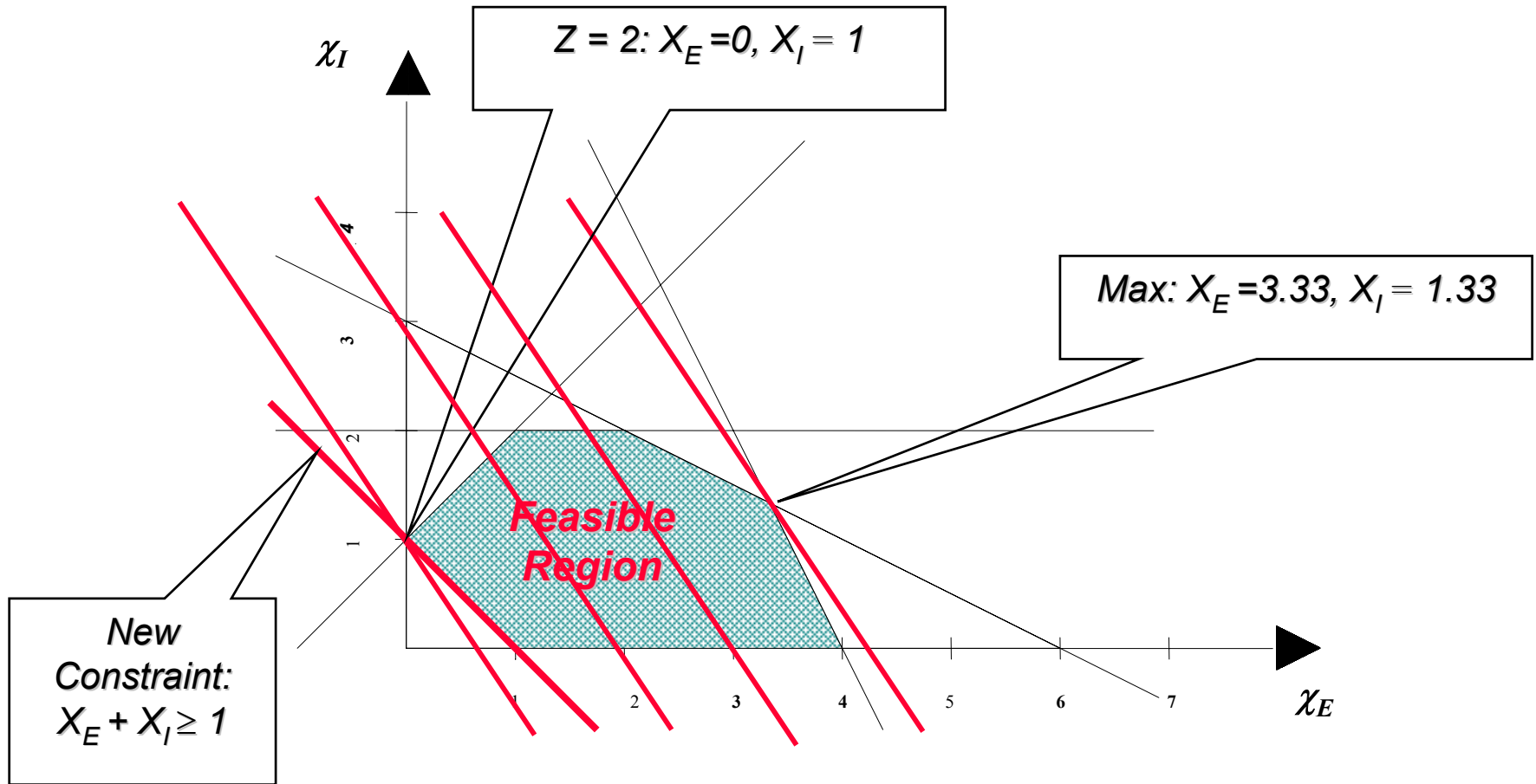
A **Corner-point feasible (CPF)** solution is a solution that lies at a corner of the feasible region.

The optimal solution is a corner point feasible solution (why?)

Team Exercise (five minutes)

- ◆ *For the Ready Mikks problem, find all the corner-point feasible solutions*
- ◆ *Suppose that another constraint is added to the problem : $X_E + X_I \geq 1$, and the problem is changed from maximization to minimization. For this new problem, find the new optimal solution*
- ◆ *Discuss and answer the following question: Is it possible to get a non-convex feasible region from the addition of a linear constraint?*

Solution



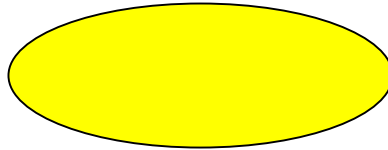
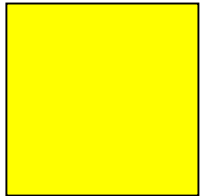
Assignment

◆ *For next class, graphically solve the “Report Strategy Problem”*

Definition of a convex set (region)

◆ *A set is convex if all the points of a line segment joining any two points in the set also fall in the set.*

◆ *Examples of convex sets*



◆ *Examples non-convex sets*

